MTH 304: Metric Spaces and Topology

Homework II

(Due 23/01)

1. Show that products and subspaces of Hausdorff spaces are Hausdorff.
2. Show that the cofinite topology in $\mathbb{R}$ is non-Hausdorff. [Hint: Find where the sequence $x_n = 1/n$ converges.]
3. Show that X is Hausdorff if and only if the *diagonal*

$$\Delta = \{(x, x) \mid x \in X\}$$

is closed in $X \times X$.

4. Let A, B, A_α be sets of a space X. Then show that
   (a) $\bar{A} \times \bar{B} = \overline{A \times B}$
   (b) $\overline{A \cup B} = \bar{A} \cup \bar{B}$.
   (c) $\cup_\alpha \bar{A}_\alpha \subset \overline{\cup_\alpha A_\alpha}$. Give an example where equality fails.
5. Let $(X, \mathcal{T})$ be a topological space, and $A \subset X$. Then the boundary of A (denoted by ∂A) is defined by

$$\partial A = \bar{A} \cap \overline{X \setminus A}.$$

   (a) Show that $\bar{A} = A^\circ \sqcup \partial A$.
   (b) Show that $\partial A = \emptyset \iff A$ is both open and closed.
   (c) $U \in \mathcal{T} \iff \partial U = \bar{U} \setminus U$.
6. For each of the following subsets $A \subset \mathbb{R}^2$, find A°, ∂A and $\bar{A}$.
   (a) $A = \mathbb{Q} \times \mathbb{R}$.
   (b) $A = \{(x, y) \mid x > 0 \text{ and } y \neq 0\}$.
   (c) $A = \{(x, y) \mid 0 < x^2 - y^2 \leq 1\}$.
   (d) $A = \{(x, y) \mid x \neq 0 \text{ and } y \leq 1/x\}$.